

ENRIQUES' CLASSIFICATION IN CHARACTERISTIC $p > 0$: THE P_{12} -THEOREM

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Dedicated to David Mumford on the occasion of his 80th birthday

ABSTRACT. The main goal of this paper is to show that Castelnuovo-Enriques' P_{12} -theorem (a precise version of the rough classification of algebraic surfaces) also holds for algebraic surfaces S defined over an algebraically closed field k of positive characteristic ($\text{char}(k) = p > 0$).

The result relies on a main theorem describing the growth of the plurigenera for properly-elliptic or properly quasi-elliptic surfaces (surfaces with Kodaira dimension equal to 1). We also discuss the limit cases, i.e. the families of surfaces which show that the results of the main theorem are sharp.

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INTRODUCTION

The main technical result of the present article, expressed in modern language, is the following one:

Main Theorem. *Let S be a projective surface of Kodaira dimension 1 defined over an algebraically closed field k , and let K_S be a canonical divisor on S , so that $\Omega_S^2 \cong \mathcal{O}_S(K_S)$.*

Then the growth of the plurigenera $P_n(S) = \dim H^0(\mathcal{O}_S(nK_S)) = \dim H^0((\Omega_S^2)^{\otimes n})$ satisfies:

- (1) $P_{12}(S) \geq 2$
- (2) *there exists $n \leq 4$ such that $P_n(S) \geq 1$*

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- (3) *there exists $n \leq 8$ such that $P_n(S) \geq 2$*
- (4) $\forall n \geq 14$ $P_n(S) \geq 2$.

While (2)-(3) of the result are new also in the classical case where k is a field of characteristic zero, (1) is due to Enriques [Enr14] in characteristic 0 and (4) was shown by Katsura and Ueno [KU85] for elliptic surfaces in all characteristics (but we reprove their result here as part of the above more general statement). Needless to say, we use in the proof of our theorem many results, lemmas and propositions previously established by many authors, especially Bombieri and Mumford, Raynaud, and Katsura-Ueno ([Mum69], [BM77], [BM76], [Ray76], [KU85]).

Most important is statement (1), which allows to extend to positive characteristic the main classification theorem of Castelnuovo and Enriques. In modern language (see the next section for more details, and a more precise and informative statement) the classification theorem says:

P_{12} -Theorem. *Let S be a projective surface defined over an algebraically closed field k .*

Then for the Kodaira dimension $\text{Kod}(S)$ we have:

- (I) $\text{Kod}(S) = -\infty \iff P_{12}(S) = 0$
- (II) $\text{Kod}(S) = 0 \iff P_{12}(S) = 1$
- (III) $\text{Kod}(S) = 1 \iff P_{12}(S) \geq 2$ and, for S minimal, $K_S^2 = 0$
- (IV) $\text{Kod}(S) = 2 \iff P_{12}(S) \geq 2$ and, for S minimal, $K_S^2 > 0$.

It should be observed that the estimates for the growth of the plurigenera are much weaker if one considers properly elliptic non algebraic surfaces, see [Ita70] who proved the analogue of (4) of the main theorem for non algebraic surfaces. Iitaka showed that, for $n \geq 86$, $H^0(\mathcal{O}_S(nK_S))$ yields the canonical elliptic fibration. One of the reasons why the estimate is much weaker depends on the failure of the Poincaré reducibility theorem, implying in the algebraic case that a certain monodromy group G is Abelian. Hence, for instance, if G is Abelian, it cannot be a Hurwitz group, i.e. G cannot have generators a, b, c of respective orders $(2, 3, 7)$ satisfying $abc = 1$.

Indeed (we omit here the simple proof) the analogue of statement (1) for non algebraic surfaces is that $P_{42} \geq 2$.

Concerning higher dimensional algebraic varieties, a natural question emerges:

Question 0.1. *Given a projective manifold of X dimension N , is there a sharp number $d = d(N)$ such that*

- (1) $\text{Kod}(X) = -\infty \iff P_d(X) = 0$
- (2) $\text{Kod}(X) = 0 \iff P_d(X) = 1$
- (3) $\text{Kod}(X) \geq 1 \iff P_d(X) \geq 2$?

Progress on a related question, about effectivity of the Iitaka fibration, was obtained, among others, by Fujino and Mori [FM00] and Birkar and Zhang [BZ16].

1. THE CLASSIFICATION THEOREM OF CASTELNUOVO AND ENRIQUES

Let S be a nonsingular projective surface defined over an algebraically closed field k , and let K_S be a canonical divisor on S , so that $\Omega_S^2 \cong \mathcal{O}_S(K_S)$. We assume

that S is minimal: this means that there does not exist an irreducible curve C of the first kind, i.e., a curve C with $C^2 = K_S \cdot C = -1$. Let us recall the definition of the basic numerical invariants associated to S , which allow its birational classification.

For each integer $m \in \mathbb{N}$, we denote as usual, following Castelnuovo and Enriques, by

$$P_m(S) := h^0(S, mK_S),$$

the m -th plurigenus of S .

In particular, the **geometric genus** is $p_g(S) := P_1(S)$, while the **arithmetic irregularity** is defined as $h(S) := h^1(\mathcal{O}_S)$, and the **arithmetic genus** is defined as

$$p_a(S) := p_g(S) - h(S) = \chi(\mathcal{O}_S) - 1.$$

To finish our comparison of classical and modern notation, recall that the **geometric irregularity** is defined as $q(S) := \frac{1}{2}b_1(S)$, where $b_1(S)$ is the first ℓ -adic Betti number of S , $b_1(S) := \dim_{\mathbb{Q}_\ell} H_{\text{et}}^1(S, \mathbb{Q}_\ell)$.

$q(S)$ is equal to the dimension of the Picard scheme $\text{Pic}^0(S)$, and also of the dual scheme $\text{Pic}^0(\text{Pic}^0(S))$, and of the Albanese variety $\text{Alb}(S) := \text{Pic}^0(\text{Pic}^0(S)_{\text{red}})$.

The above numbers are all equal in characteristic zero: $q(S) = h(S) = h^0(\Omega_S^1)$, but not in characteristic $p > 0$, where one just has some inequalities.

Since $H^1(\mathcal{O}_S)$ is the Zariski tangent space to the Picard scheme at the origin ([Mum66]), one has the inequalities (cf. [BM77])

$$h(S) \geq q(S), \quad 2p_g(S) \geq \Delta := 2(h(S) - q(S)) = 2h(S) - b_1(S) \geq 0.$$

The inequality $h^0(\Omega_S^1) \geq q$ was shown by Igusa [Igu55-1], and there are examples where equality does not hold, cf. [Igu55-2], [Mum61]¹.

Moreover, the **linear genus** $p^{(1)}(S) := K_S^2 + 1$ is the arithmetic genus of any canonical divisor on the minimal surface. It is a birational invariant for every non ruled algebraic surface.

The classification of smooth projective curves C is given in terms of the genus $g(C) := h^0(\mathcal{O}_C(K_C))$,

- (I) $g(C) = 0 \iff C \cong \mathbb{P}^1$.
- (II) $g(C) = 1 \iff \mathcal{O}_C \cong \mathcal{O}_C(K_C) \iff C \text{ is an elliptic curve (it is isomorphic to a plane cubic curve).}$
- (III) $g(C) \geq 2 \iff C \text{ is of general type, i.e., } H^0(C, \mathcal{O}_C(mK_C)) \text{ yields an embedding of } C \text{ for all } m \geq 3.$

Enriques and Castelnuovo ([Enr14] and [CE15]) were able to give the surface classification essentially in terms of $P_{12}(S)$, as follows:

Theorem 1.1. (P_{12} -theorem of Castelnuovo-Enriques)

¹The space of regular one forms on the Albanese variety A pulls back injectively to a subspace V of the space $H^0(\Omega_S^1)$, contained in the space of d -closed forms; it is an open question how to characterize V , for instance Illusie suggested V could be the intersection of the kernels of $d \circ C^m$, where C is the Cartier operator, and m is any positive integer: see [Ses60], [DI87], [IR83]

Let S be a projective smooth surface defined over an algebraically closed field k of characteristic zero, and let $p^{(1)}(S) := K_S^2 + 1$ be the linear genus of a minimal model in the birational equivalence class of S . Then

- (I) $P_{12}(S) = 0 \iff S$ is ruled $\iff S$ is birational to a product $C \times \mathbb{P}^1$, $g(C) = q(S) = h(S)$.
- (II) $P_{12} = 1 \iff \mathcal{O}_S \cong \mathcal{O}_S(12K_S)$.
- (III) $P_{12} \geq 2$ and $p^{(1)}(S) = 1 \iff S$ is properly elliptic, i.e. $H^0(S, \mathcal{O}_S(12K_S))$ yields a fibration over a curve with general fibres elliptic curves.
- (IV) $P_{12} \geq 2$ and $p^{(1)}(S) > 1 \iff S$ is of general type, i.e. $H^0(S, \mathcal{O}_S(mK_S))$ yields a birational embedding of S for m large ($m \geq 5$ indeed suffices, as conjectured by Enriques in [Enr49] and proven by Bombieri [Bom73]).

Moreover, if S is minimal, then in modern terminology:

- **Case (I):** $S \cong \mathbb{P}^2$ or S is a \mathbb{P}^1 -bundle over a curve C ,
- **Case (II),** $p_g(S) = 1, q(S) = 2 \iff \mathcal{O}_S \cong \mathcal{O}_S(K_S), q(S) = 2 \iff S$ is an Abelian surface.
- **Case (II),** $p_g(S) = 1, q(S) = 0 \iff \mathcal{O}_S \cong \mathcal{O}_S(K_S), q(S) = 0 \iff S$ is a K3 surface.
- **Case (II),** $p_g(S) = 0, q(S) = 0 \iff \mathcal{O}_S \not\cong \mathcal{O}_S(K_S), \mathcal{O}_S \cong \mathcal{O}_S(2K_S), q(S) = 0 \iff S$ is an Enriques surface.
- **Case (II),** $q(S) = 1 (\Rightarrow p_g(S) = 0) \iff \mathcal{O}_S \not\cong \mathcal{O}_S(K_S), \mathcal{O}_S \cong \mathcal{O}_S(mK_S), m \in \{2, 3, 4, 6\}, q(S) = 1 \iff S$ is a hyperelliptic surface.
- **Case (III),** $p_a(S) = -1 \iff S \cong C \times E, g(E) = 1, g(C) = q(S) - 1$.

A modern account of the Castelnuovo-Enriques classification of surfaces was first given in [Steklov 65] and in [Kod68], then it appeared also in [BH75], [Bea78] (this is the only text which mentions the P_{12} -theorem, in the historical note on page 118), later also in [Bad81] and [BPV84].

Remark 1.2. i) Nowadays, cases (I)-(IV) are distinguished according to the Kodaira dimension, which is defined to be $-\infty$ if all the plurigenera vanish ($P_n = 0 \forall n \geq 1$), otherwise it is defined as the maximal dimension of the image of some n -pluricanonical map (the map associated to $H^0(\mathcal{O}_X(nK_X))$).

ii) The occurrence of the number 12 is rather miraculous: it first appears since, by the canonical divisor formula 1.6, in case (II) the equation

$$2 = \sum_j (1 - \frac{1}{m_j})$$

admits only the following (positive) integer solutions:

$$(2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6)$$

and then we get a set of integers m_j whose least common multiple is precisely 12.

According to the several cases we have $2K_S \equiv 0, 3K_S \equiv 0, 4K_S \equiv 0, 6K_S \equiv 0$, where $D \equiv 0$ means that F is linearly equivalent to zero, i.e. $\mathcal{O}_S(D) \cong \mathcal{O}_S$. It follows that in case (II) we have $12K_S \equiv 0$, hence $P_{12} = 1$.

The second occurrence is more subtle, and is the heart of the theorem: in case (III) one has $P_{12} \geq 2$.

It is now customary (the name 'key theorem' is due to [Bea78]) to see the two major steps of surface classification as follows:

Theorem 1.3. (Key Theorem) *If S is minimal, then*

K_S is nef (i.e. , $K_S \cdot C \geq 0$ for all curves $C \subset S$) $\iff S$ is nonruled.

Theorem 1.4. (Crucial Theorem) *S is minimal, with $p_g(S) = 0, q(S) = 1$*

*$\iff S$ is **isogenous** to an elliptic product, i.e. S is the quotient $(C_1 \times C_2)/G$ of a product of curves of genera*

$$g_1 := g(C_1) = 1, g_2 := g(C_2) \geq 1,$$

by a free action of a finite group of product type (G acts faithfully on C_1, C_2 and we take the diagonal action $g(x, y) := (gx, gy)$), and such that moreover if we denote by $g'_j = g(C_j/G)$, then $g'_1 + g'_2 = 1$.

More precisely, let A be the Albanese variety of S , which is an elliptic curve and let

$$\alpha : S \rightarrow A$$

be the Albanese map.

Then either:

1) S is a hyperelliptic surface, $(C_1 \times C_2)/G$, $g_2 = 1$, G is a subgroup of translations of C_1 , $A = C_1/G$, while $C_2/G \cong \mathbb{P}^1$.

In this case all the fibres of the Albanese map are isomorphic to C_2 , $P_{12}(S) = 1$ and S admits also an elliptic fibration $\psi : S \rightarrow C_2/G \cong \mathbb{P}^1$.

2) S is properly elliptic ($P_{12}(S) \geq 2$) and the genus $g = g_2$ of the Albanese fibres satisfies $g_2 \geq 2$: again G is a subgroup of translations of C_1 , $A = C_1/G$, $C_2/G \cong \mathbb{P}^1$, all the fibres of the Albanese map are isomorphic to C_2 .

3) S is properly elliptic ($P_{12}(S) \geq 2$) and the genus $g = g_1$ of the Albanese fibre satisfies $g_1 = 1$: $A = C_2/G$, $C_1/G \cong \mathbb{P}^1$, and the fibres of the Albanese map

$$\alpha : S = (C_1 \times C_2)/G \rightarrow A = C_2/G$$

are either isomorphic to the elliptic curve C_1 or are multiples of a smooth elliptic curve isogenous to C_1 .

Remark 1.5. A crucial observation, used by Enriques in [Enr14] for the P_{12} -theorem is that in the first two cases the group G is Abelian. The crucial ingredient is the canonical divisor (canonical bundle) formula, established by Enriques, Kodaira, and then extended to positive characteristic by Bombieri and Mumford.

Theorem 1.6. [BM77, p.27 Theorem 2.] *Let $f : S \rightarrow C$ be a relatively minimal fibration such that the arithmetic genus of a fibre equals 1 (the general fibre is necessarily smooth elliptic in characteristic zero, but it can be rational with one cusp in characteristic 2, 3: the latter is called the quasi-elliptic case).*

Let $\{q_1, \dots, q_r\} \subset C$ the set of points over which the fibre $f^{-1}(q_i) = m_i F'_i$ is a multiple fibre (i.e. $m_i \geq 2$ and F'_i is not a multiple of any proper sub-divisor), and consider the coherent sheaf $R^1 f_(\mathcal{O}_S)$ on the smooth curve C , which decomposes as*

$$R^1 f_*(\mathcal{O}_S) = \mathcal{O}_C(L) \oplus T,$$

where $\mathcal{O}_C(L)$ is an invertible sheaf and T is a torsion subsheaf with $\text{supp}(T) \subset \{q_1, \dots, q_r\}$. The fibres over the points of $\text{supp}(T)$ are called **wild fibres**, moreover $T = 0$ if $\text{char}(k) = 0$.

Then

$$K_S = f^*(\delta) + \sum_{i=1}^r a_i F'_i, \delta := -L + K_C$$

where

- (i) $0 \leq a_i < m_i$;
- (ii) $a_i = m_i - 1$ if $m_i F'_i$ is not wild (i.e., $q_i \notin \text{supp}(T)$);
- (iii) $d := \deg(\delta) = \deg(-L + K_C) = 2g(C) - 2 + \chi(\mathcal{O}_S) + \text{length}(T)$,
where $g(C)$ is the genus of C .

Let us see how the above applies in characteristic zero and in the special subcase: $p_g = 0, q = 1$, genus of the Albanese fibres equal 1, there exist multiple fibres.

Then, for $n = 2$, since we have $\deg(\delta) = 0$, follows that

$$2K_S = \sum_{i=1}^r (m_i - 2)F'_i + f^*(\delta + \sum_{i=1}^r q_i).$$

The divisor $\delta + \sum_{i=1}^r q_i$ is effective by the Riemann Roch theorem on the elliptic curve A , so we have written $2K_S$ as the sum of two effective divisors.

Hence we obtain that $P_2 \geq 1$, and similarly one gets that $P_{12} \geq 6$.

2. THE P_{12} -THEOREM IN POSITIVE CHARACTERISTIC

The extension of the Castelnuovo-Enriques classification of surfaces to the case of positive characteristic was achieved by D. Mumford and E. Bombieri (cf. [BM76, Section 3], [BM77, Theorem 1.]).

In a remarkable series of three papers they got most of the following full result.

Theorem 2.1. (P_{12} -theorem in positive characteristic)

Let S be a projective smooth surface defined over an algebraically closed field k of characteristic $p > 0$, and let $p^{(1)}(S) := K_S^2 + 1$ be the linear genus of a minimal model in the birational equivalence class of S . Then

- (I) $P_{12}(S) = 0 \iff S$ is ruled $\iff S$ is birational to a product $C \times \mathbb{P}^1$,
 $g(C) = q(S) = h(S)$.
- (II) $P_{12} = 1 \iff \mathcal{O}_S \cong \mathcal{O}_S(12K_S)$.
- (III) $P_{12} \geq 2$ and $p^{(1)}(S) = 1 \iff S$ is properly elliptic or properly quasi-elliptic, i.e. $H^0(S, \mathcal{O}_S(12K_S))$ yields a fibration over a curve with general fibres either elliptic curves or rational curves with one cusp.
- (IV) $P_{12} \geq 2$ and $p^{(1)}(S) > 1 \iff S$ is of general type, i.e. $H^0(S, \mathcal{O}_S(mK_S))$ yields a birational embedding of S for m large (indeed, $m \geq 5$ suffices).

Moreover, Bombieri and Mumford in [BM77] and [BM76] gave a full description of the surfaces in the classes (I) and (II) (with new non classical surfaces), but classes (II) and (III) were not distinguished by the behaviour of the 12-th plurigenus, but only by the Kodaira dimension, i.e., by the growth of $P_n(S)$ as $n \rightarrow \infty$.

The sharp statement ($\forall m \geq 5$) in case (IV), established by Bombieri [Bom73, Main Theorem] in characteristic zero, was extended by T. Ekedahl's to the case of positive characteristic (cf. [Eke88, Main Theorem], see also [CF93] and [CFHR] for a somewhat simpler proof).

3. AUXILIARY RESULTS AND PROOF OF THE P_{12} -THEOREM

Case (III) can be divided into two subcases: properly elliptic fibrations and properly quasi-elliptic fibrations.

Recall the definition of quasi-elliptic surfaces:

Definition 3.1. *A quasi-elliptic surface S is a nonsingular projective surface admitting a fibration $f : S \rightarrow C$ over a nonsingular projective curve C such that $f_*\mathcal{O}_X = \mathcal{O}_C$ and such that the general fibres of f are rational curves with one cusp. If the fibration f is induced by $H^0(S, \mathcal{O}_S(nK_S))$ for some $n > 0$, we call S a properly quasi-elliptic surface.*

Remark 3.2. *1) By a result of J. Tate (cf. [Tat52]), quasi-elliptic fibrations only appear in characteristic 2 and 3.*

2) in case (III), where $P_n(S) := \dim H^0(S, \mathcal{O}_S(nK_S))$ grows linearly with n , S is necessarily properly elliptic or properly quasi-elliptic.

The case where S admits a properly elliptic fibration was treated by T. Katsura and K. Ueno who proved in [KU85, Theorem 5.2.] that for any properly elliptic surface S , $\forall m \geq 14$, $P_m(S) \geq 2$ and showed the existence of an example where $P_{13} = 1$. They show that the situation is essentially the same as in characteristic zero. The fact that $P_{12}(S) \geq 2$ follows from our more general theorem, which uses several auxiliary results developed by Raynaud and Katsura-Ueno (they will be recalled in the sequel).

Theorem 3.3. (Main Theorem) *Let $f : S \rightarrow C$ be a properly elliptic or quasi-elliptic fibration. Then*

- (1) $P_{12}(S) \geq 2$
- (2) *there exists $n \leq 4$ such that $P_n(S) \neq 0$*
- (3) *there exists $n \leq 8$ such that $P_n(S) \geq 2$*
- (4) $\forall n \geq 14$ $P_n(S) \geq 2$.

Remark 3.4. *Let us indicate the examples (see remark 4.2) which show that in theorem 3.3 the inequalities in our assumptions are the best possible.*

(2) and (4): in the notation of (2) of theorem 1.4 we let $G = \mathbb{Z}/2 \oplus \mathbb{Z}/6$; clearly G is isomorphic to a subgroup of any elliptic curve. In order to obtain a curve C_2 with a G action such that $C_2/G \cong \mathbb{P}^1$ we consider a G -Galois covering C_2 of \mathbb{P}^1 branched in 3 points, and with local monodromies

$$(1, 0), (0, 1), (-1, -1).$$

This exists, by Riemann's existence theorem since the sum of the three local monodromies equals zero. This example yields the curve C_2 with affine equation $y^2 = x^6 - 1$, which is smooth in characteristic $\neq 2, 3$, see [KU85].

The fibration $f : S \rightarrow C_2/G \cong \mathbb{P}^1$ is elliptic and has exactly three singular fibres, multiple with multiplicities 2, 6, 6. It follows that

$$P_n(S) = -2n + 1 + [n/2] + 2 \cdot [5n/6],$$

where $[a]$ denotes the integral part of a .

Follows that $P_1 = P_2 = P_3 = 0$, $P_4 = P_5 = 1$, $P_6 = 2$, $P_{13} = 1$.

(2) and (3) : in the notation of (2) of theorem 1.4 we let $G = \mathbb{Z}/10$; clearly G is isomorphic to a subgroup of any elliptic curve. In order to obtain a curve C_2 with a G action such that $C_2/G \cong \mathbb{P}^1$ we consider a G -Galois covering C_2 of \mathbb{P}^1 branched in 3 points, and with local monodromies

$$(5), (4), (1).$$

This exists, by Riemann's existence theorem since the sum of the three local monodromies equals zero. Indeed, the curve is defined by the affine equation $y^2 = x^5 - 1$ and is smooth in characteristic $\neq 2, 5$.

The fibration $f : S \rightarrow C_2/G \cong \mathbb{P}^1$ is elliptic and has exactly three singular fibres, multiple with multiplicities 2, 5, 10. It follows that

$$P_n(S) = -2n + 1 + [n/2] + [4n/5] + [9n/10].$$

Follows that $P_1 = P_2 = P_3 = 0$, $P_4 = P_5 = P_6 = P_7 = 1$, $P_8 = P_9 = 2$, $P_{10} = 3$, $P_{11} = 1$, $P_{12} = P_{13} = 2$.

In the case of properly quasi-elliptic fibrations, we shall use some result of Raynaud, [Ray76], and a corollary developed by Katsura and Ueno (lemmas 2.3 and 2.4 [KU85]).

Given a multiple fibre mF' we denote by $\omega_n := \mathcal{O}_{nF'}(K_S + nF')$ the dualizing sheaf of nF' .

Observe that F' is an indecomposable divisor of elliptic type, hence (see [Mum69] [CFHR]) for any degree zero divisor L on F' , we have $h^0(\mathcal{O}_{F'}(L)) = h^1(\mathcal{O}_{F'}(L))$, and these dimensions are either $= 0$, or $= 1$, the latter case occurring if and only if $\mathcal{O}_{F'}(L) \cong \mathcal{O}_{F'}$.

Consider now the exact sequence

$$0 \rightarrow \mathcal{O}_{F'}(-(n-1)F') \rightarrow \mathcal{O}_{nF'} \rightarrow \mathcal{O}_{(n-1)F'} \rightarrow 0,$$

and apply the previous remark for $L = -(n-1)F'$ to infer that

$$h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'}) \text{ or } = h^0(\mathcal{O}_{(n-1)F'}) + 1,$$

the second possibility occurring only if

$$(**) \quad \mathcal{O}_{F'}((n-1)F') \cong \mathcal{O}_{F'}.$$

Conversely, if $(**)$ holds, either $h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'})$ and $h^1(\mathcal{O}_{nF'}) = h^1(\mathcal{O}_{(n-1)F'})$, or both h^0, h^1 grow by 1 for nF' .

This in any case shows that the function $h^0(\mathcal{O}_{nF'})$ is monotone nondecreasing. One says that n is a **jumping value** if $n \geq 1$ and $h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'}) + 1$. Considering all the $n \geq 1$, we can then define the first jumping value, the second, and so on (they are then clearly ≥ 2).

Recall now:

Proposition 3.5 ([BM77] Proposition 4 and [Ray70] Proposition 6.3.5.). *Since $(\mathcal{O}_{F'_i}(F'_i))$ is a torsion element in the Picard group of F'_i , we consider its torsion order:*

$$\nu_i := \text{order}(\mathcal{O}_{F'_i}(F'_i)).$$

We have then

- (1) ν_i divides both m_i and $a_i + 1$;
- (2) letting $p = \text{char}(k)$, there exists an integer $e_i \geq 1$ such that $m_i = \nu_i \cdot p^{e_i}$;
- (3) $h^0(F'_i, \mathcal{O}_{(\nu_i+1)F'_i}) \geq 2$, $h^0(F_i, \mathcal{O}_{\nu_i F'_i}) = 1$, so that $\nu_i + 1$ is a jumping value;
- (4) $h^0(F'_i, \mathcal{O}_{rF'_i})$ is a non-decreasing function of r .

Using 3.5, we get the following corollary

Corollary 3.6 ([BM77], Corollary.). *If $h^1(S, \mathcal{O}_S) \leq 1$, we have either*

$$a_i = m_i - 1$$

or

$$a_i = m_i - 1 - \nu_i.$$

More precise results are the following two lemmas of M. Raynaud (cf. [Ray76], [BT14, Section 2]).

Lemma 3.7. [Ray76, Corollaire 3.7.6.] *Let $f : S \rightarrow C$ be an elliptic or quasi-elliptic fibration with $f^{-1}(q) = mF'$ a multiple fibre over $q \in C$. Then for any integer $n \geq 2$:*

- (i) *The dualizing sheaf $\omega_n := \mathcal{O}_{nF'}(K_S + nF')$ of nF' is non-trivial iff $h^0(\omega_n) = h^0(\omega_{n-1})$.*
- (ii) *ω_n is trivial iff $h^0(\omega_n) = h^0(\omega_{n-1}) + 1$.*

Lemma 3.8. [Ray76, Lemma 3.7.7.] *Notation being as in Lemma 3.7, observe that the invertible sheaves $\mathcal{O}_{nF'}(F')$ are torsion elements in the Picard group of nF' . There are only two possibilities for their torsion orders. Setting $o_n := \text{Ord}(\mathcal{O}_{nF'}(F'))$ (hence $o_1 = \nu$), we have*

- (i) $o_n = o_{n-1}$;
- (ii) $o_n = p \cdot o_{n-1}$.

Moreover, case (ii) occurs only if ω_n is trivial.

Proof. Setting $\mathfrak{N} := \mathcal{O}_{F'}(-(n-1)F')$, we consider the following two exact sequences:

$$(1) \quad 0 \rightarrow \mathfrak{N} \rightarrow \mathcal{O}_{nF'} \rightarrow \mathcal{O}_{(n-1)F'} \rightarrow 0.$$

$$(2) \quad 0 \rightarrow 1 + \mathfrak{N} \rightarrow \mathcal{O}_{nF'}^* \rightarrow \mathcal{O}_{(n-1)F'}^* \rightarrow 0.$$

Since $\mathfrak{N}^2 = 0$, the map $x \mapsto 1 + x$ defines an isomorphism of abelian sheaves:

$$\beta : \mathfrak{N} \simeq 1 + \mathfrak{N}.$$

Taking the induced long exact sequence of (1) and (2) and observing that $H^2(F', \mathfrak{N}) \simeq H^2(F', 1 + \mathfrak{N}) = 0$, we get

$$(3) \quad H^0(\mathcal{O}_{(n-1)F'}) \xrightarrow{\partial} H^1(\mathfrak{N}) \rightarrow H^1(\mathcal{O}_{nF'}) \xrightarrow{\alpha} H^1(\mathcal{O}_{(n-1)F'}) \rightarrow 0$$

and

$$(4) \quad H^0(\mathcal{O}_{(n-1)F'}^*) \xrightarrow{\partial^*} H^1(1 + \mathfrak{N}) \rightarrow \text{Pic}(nF') \xrightarrow{\alpha^*} \text{Pic}((n-1)F') \rightarrow 0.$$

By a result of F. Oort (cf. [Oor62, §6]), we have that $H^1(\beta)(\text{Im}(\partial)) = \text{Im}(\partial^*)$.

Since $H^1(\mathfrak{N})$ is a $\mathbb{Z}/p\mathbb{Z}$ -vector space, we see that any element in $\ker(\alpha^*)$ has p -th power equal to 1, hence we have $o_n = o_{n-1}$ or $o_n = po_{n-1}$.

If $o_n = po_{n-1}$, then $\ker(\alpha^*) \neq \{1\}$ and hence $\ker(\alpha) \neq \{0\}$. Since $h^1(nF', \mathcal{O}_{nF'}) = h^0(\omega_n)$, by lemma 3.7 we have that $h^0(\omega_n) = h^0(\omega_{n-1}) + 1$ and ω_n is trivial. \square

Assume that we have a multiple fibre over the point q_j , and denote by t_j the length of the skyscraper sheaf T at q_j .

Then, by the base change theorem we have

$$t_j + 1 = \text{rk}_{q_j} \mathcal{R}^1 f_*(\mathcal{O}_S) = h^1(\mathcal{O}_{m_j F'_j}) = h^0(\mathcal{O}_{m_j F'_j}).$$

The two lemmas by Raynaud imply the following very useful corollary, which holds more generally also for quasi-elliptic fibrations.

Corollary 3.9. [Ray76, Lemma 3.7.9] [KU85, Lemmas 2.3, 2.4] (1) Let $n_j^{(i)}$ be the i -th jumping value of a wild fibre mF'_j (recall that $n_j^{(i)} \geq 2$).

Setting $\nu_j := \text{Ord}(\mathcal{O}_{F'_j}(F'_j))$, we have

$$n_j^{(1)} = \nu_j + 1,$$

and

$n_j^{(2)} = 2\nu_j + 1$ if $\text{Ord}(\mathcal{O}_{(\nu_j+1)F'_j}) = \nu_j$, or $= (p+1)\nu_j + 1$ if $\text{Ord}(\mathcal{O}_{(\nu_j+1)F'_j}) = p\nu_j$.

(2) If $h^0(\mathcal{O}_{mF'_j}) = 2 \Leftrightarrow t_j = 1$, then the contribution $a_j F'_j$ to the canonical divisor formula satisfies $a_j = m_j - 1$ or $a_j = m_j - 1 - \nu_j$.

(3) If $h^0(\mathcal{O}_{mF'_j}) = 3 \Leftrightarrow t_j = 2$, then $a_j = m_j - 1$ or $a_j = m_j - 1 - \nu_j$ or $a_j = m_j - 1 - 2\nu_j$ or $a_j = m_j - 1 - (p+1)\nu_j$.

Finally, Katsura and Ueno proved for elliptic fibrations in characteristic p the analogue of a result which in characteristic zero follows from the description of the fundamental group of the complement of a finite set of points in \mathbb{P}^1 .

Definition 3.10. [[KU85], Definition 3.1.] Let $f : S \rightarrow \mathbb{P}^1$ be an elliptic fibration with $\chi(S, \mathcal{O}_S) = 0$, let $m_i F'_i$, $i = 1, \dots, k$, be the multiple fibres, and let as usual ν_i be the torsion order of $\mathcal{O}_{F'_i}(F'_i)$.

Then S is said to be of type $(m_1, \dots, m_r | \nu_1, \dots, \nu_r)$.

Definition 3.11. [[KU85], Definition 3.2.] Given $1 \leq i \leq r$, we say that two sequences $(m_1, \dots, m_r | \nu_1, \dots, \nu_r)$ satisfy condition U_i , if there do exist integers n_1, \dots, n_r (depending on i) such that

- $n_i \equiv 1 \pmod{\nu_i}$ and

- $\sum_{j=1}^r n_j/m_j \in \mathbb{Z}$.

Theorem 3.12 ([KU85], Theorem 3.3.). *In the situation of 3.10, then the sequences $(m_1, \dots, m_r | \nu_1, \dots, \nu_r)$ satisfy condition $U_i \forall i = 1, 2, \dots, r$.*

4. PROOF OF THE MAIN THEOREM 3.3

Let $f : S \rightarrow C$ be a relatively minimal properly elliptic or properly quasi-elliptic fibration. Set here $g := g(C)$ and set $t = \text{length}(T)$, where T is the torsion sheaf appearing in the canonical bundle formula.

The first important observation is that in the canonical bundle formula the term $\chi(\mathcal{O}_S)$ is ≥ 0 , by Mumford's extension of Castelnuovo's theorem ([Mum69]).

The case $\chi(\mathcal{O}_S) + t \geq 1$, $g \geq 1$ is quickly disposed of by observing that

$$P_n(S) = h^0(\mathcal{O}_S(nK_S)) \geq h^0(\mathcal{O}_C(n\delta)) \geq g + (n-1) \geq n.$$

If $g \geq 2$, and $\chi(\mathcal{O}_S) = t = 0$, we are done, since then $P_n \geq (2n-1)(g-1)$.

If instead $g = 1$, $\chi(\mathcal{O}_S) = t = 0$ there are no wild fibres, and since the canonical divisor is not numerically trivial, $\sum_j (1 - \frac{1}{m_j}) > 0$, hence

$$nK_S = \sum_j n(m_j - 1)F'_j = \sum_j \left[\frac{n(m_j - 1)}{m_j} \right] F_j + m_j \left\{ \frac{n(m_j - 1)}{m_j} \right\} F'_j,$$

so we can rewrite

$$nK_S = \sum_j f^* \left(\left[\frac{n(m_j - 1)}{m_j} \right] q_j \right) + D,$$

where D is an effective divisor (with integral coefficients).

Hence

$$(*) \quad P_n(S) \geq \sum_j \left[\frac{n(m_j - 1)}{m_j} \right] \geq \left[\frac{n}{2} \right].$$

We may therefore assume that $g = 0$.

For $g = 0$, if $\chi(\mathcal{O}_S) + t \geq 3$, we get $P_n(S) \geq n + 1$.

If $g = t = 0$, $\chi(\mathcal{O}_S) = 2$, then again there are no wild fibres and the same argument as in (*) yields

$$P_n(S) \geq 1 + \sum_j \left[\frac{n(m_j - 1)}{m_j} \right] \geq 1 + \left[\frac{n}{2} \right].$$

We are left with the following possibilities:

Case (1) $\chi(\mathcal{O}_S) = 1$, $t = 1$ and $g = 0$;

Case (2) $\chi(\mathcal{O}_S) = 0$, $t = 2$ and $g = 0$;

Case (3) $\chi(\mathcal{O}_S) = 0$, $t = 1$ and $g = 0$;

Case (4) $\chi(\mathcal{O}_S) = t = 0$ and $g = 0$.

The next lemma shows that, except possibly in case (1), we need to take care only of the properly elliptic case.

Lemma 4.1. *There exists no quasi-elliptic fibration $f : S \rightarrow \mathbb{P}^1$ with $\chi(\mathcal{O}_S) = 0$.*

Proof. Assume we have such a fibration.

Let $\alpha : S \rightarrow A$ be the Albanese map of S and assume that $q := \dim(A) \geq 1$. Since a general fibre of f is a cuspidal rational curve, whose image in A must be a single point, we see that α factors through f . Hence the image of α is a point: since the image generates A , A is a point and $q = 0$, a contradiction.

We conclude that $q = 0$, hence $p_g \geq h$ and $\chi(\mathcal{O}_S) \geq 1$, a contradiction. \square

Let us now proceed with the proof.

We can write

$$K_S \equiv dF + \sum_i a_i F'_i,$$

where F is a fibre of f . We observe that $p_g(S) = \max(0, d+1)$.

Indeed, if $p_g \geq 1$, we can write $|K_S| = |M| + \Phi$, where Φ is the fixed part, and where the movable part is of the form $(p_g - 1)F$.

Hence K_S is linearly equivalent to an effective divisor D of the form $K_S \equiv (p_g - 1)F + \sum_i b_i F'_i$, with $0 \leq b_i < m_i$.

If $d \geq 0$, then $p_g = d + 1$ and the fixed part $\Phi = \sum_i a_i F'_i$.

Otherwise, if $d < 0$, and we assume $p_g \geq 1$ we have a linear equivalence of effective divisors: $(|d| + p_g - 1)F + \sum_i b_i F'_i \equiv \sum_i a_i F'_i$ which shows that $|d| + p_g - 1 = 0$, a contradiction.

Hence in our cases we have respectively:

Case (1) $\chi(\mathcal{O}_S) = 1$, $t = 1$, $h = 1$, $p_g = 1$ and $g = 0$;

Case (2) $\chi(\mathcal{O}_S) = 0$, $t = 2$, $h = 2$, $p_g = 1$ and $g = 0$;

Case (3) $\chi(\mathcal{O}_S) = 0$, $t = 1$, $h = 1$, $p_g = 0$ and $g = 0$;

Case (4) $\chi(\mathcal{O}_S) = t = 0$, $h = 1$, $p_g = 0$ and $g = 0$.

Observe therefore that corollary 3.6 applies in all cases except (2).

Case (1): $K_S \equiv \sum_i a_i F'_i$, and if there exists a multiple fibre for which $a_j = m_j - 1$, we are done, since then $P_n \geq [n/2] + 1$.

Otherwise, there is exactly one multiple fibre, wild, with $t_j = 1$, and by proposition 3.6 and proposition 3.5 $a := a_j$ satisfies

$$a = m - 1 - \nu = \nu(p^e - 1) - 1 > 0.$$

If $\nu = 1$, we obtain $a/m = \frac{m-2}{m} \geq 1/3$, if $\nu \geq 2$ then we get

$$a/m \geq \frac{p^e - 1 - 1/2}{p^e} = \frac{2p^e - 3}{2p^e} \geq 1/4$$

and accordingly $P_n \geq [n/3] + 1$, $P_n \geq [n/4] + 1$. \square

Case (2): Again $K_S \equiv \sum_i a_i F'_i$, and if there exists a multiple fibre for which $a_j = m_j - 1$, we are done, since then $P_n \geq [n/2] + 1$.

Otherwise there are only wild fibres, either one with $t_1 = 2$, or two with $t_1, t_2 = 1$. In the latter case by corollary 3.9 we have $a_j = m_j - 1 - \nu_j$, and we argue as in case (1).

In the former case we are left (set $m := m_1, a := a_1, \nu := \nu_1$) with the cases $a = m - 1 - 2\nu$ or $a = m - 1 - (p + 1)\nu$. It is clear that the first possibility will give a better estimate than the second, hence we treat the second.

Here

$$\frac{a}{m} = \frac{p^e - p - 1 - 1/\nu}{p^e}$$

which is a monotone increasing function of e, ν, p .

We must have $e \geq 2$, and for $e = 2, \nu = 1$ we must have $p \geq 3$.

In conclusion, for $\nu = 1$, $\frac{a}{m} \geq \min(\frac{4}{9}, \frac{4}{8}) = \frac{4}{9} \Rightarrow P_n \geq [\frac{4n}{9}] + 1$.

Instead for $\nu \geq 2$, the minimum is for $p = 2, e = 2, \nu = 2$, and we obtain $\frac{a}{m} \geq \frac{1}{8}$.

In this case we get $P_n = [n/8] + 1$, which would be a limit case, but the actual existence of this case with only one multiple fibre, and the above numerical characters, is unclear to us.

□

Case (3): Here $K_S \equiv -F + \sum_i^r a_i F'_i$, where F is a fibre of f . Since some multiple of K_S is linearly equivalent to an effective divisor, we have

$$(*) \quad -1 + \sum_i^r \frac{a_i}{m_i} > 0,$$

and it follows that $r \geq 2$. Since $t = 1$, there exists one and only one wild fibre, say $m_1 F'_1$, with $t_1 = 1$: by proposition 3.6 and proposition 3.5 $a_1 = m_1 - 1$, or $a_1 = m_1 - 1 - \nu_1$. Hence we can rewrite K_S as follows:

$$K_S \equiv -F + a_1 F'_1 + \sum_{i=2}^r (m_i - 1) F'_i,$$

so that

$$P_n = \max(0, 1 - n + [\frac{na_1}{m_1}] + \sum_{i=2}^r [\frac{n(m_i - 1)}{m_i}]).$$

If $r \geq 4$ or $r = 3, a_1 = m_1 - 1$, we have $P_n \geq 1 - n + 3[n/2]$, and writing $n = 2k + s, s \in \{0, 1\}$, we get $P_n \geq 1 - 2k - s + 3k = 1 + k - s$, which is at least 1 for $n \geq 2$, and ≥ 2 for $n \geq 4$.

In the case where $r = 3, a_1 = m_1 - 1 - \nu_1$, consider first the possibility $a_1 = 0$.

Then $(*)$ implies that m_2 or $m_3 \geq 3$, and we get

$$P_n \geq 1 - n + [2n/3] + [n/2].$$

Writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$P_n \geq 1 - 2k - s + k + [(k + 2s)/3] + k = 1 + [(k + 2s)/3] - s,$$

which is $1 + [k/3]$ if $s = 0$ and $[(k + 2)/3]$ when $s = 1$. Hence we get $P_n \geq 1$ for $n \geq 2, P_6 \geq 2$ and $P_n \geq 2$ for $n \geq 8$.

If instead $a_1 > 0$, we are of course done if m_2 or m_3 is ≥ 3 . The remaining case is $m_2 = m_3 = 2$, and now condition U_1 implies that there exists an integer l such that $2(l\nu_1 + 1)/(p^{e_1}\nu_1) \in \mathbb{Z}$, which implies $\nu_1 | 2$. Therefore we conclude that

$$\frac{a_1}{m_1} = \frac{m_1 - 1 - \nu_i}{m_1} \geq \frac{m_1 - 3}{m_1},$$

whence $m_1 \geq 4$ and $a_1/m_1 \geq 1/4$. Hence we have

$$P_n \geq 1 - n + [n/4] + 2[n/2].$$

We get $P_n \geq 1 + [k/2]$ for even $n = 2k$ and $P_n \geq [(2k+1)/4]$ for odd $n = 2k+1$. Hence we have $P_2 \geq 1$, $P_4 \geq 2$ and $P_n \geq 2$ for $n \geq 8$.

We are left with the case $r = 2$.

Assume first that $a_1 = m_1 - 1$: the situation is then identical to the case $r = 3$, $a_1 = 0$, and we are done.

We may therefore assume that $a_1 = m_1 - 1 - \nu_1 > 0$, and inequality $(*)$ becomes now

$$(**) \quad 1 - \frac{1 + 1/\nu_1}{p^{e_1}} - \frac{1}{m_2} > 0,$$

and we have

$$P_n \geq 1 - n + \left[\frac{n(p^{e_1} - 1 - 1/\nu_1)}{p^{e_1}} \right] + \left[\frac{n(m_2 - 1)}{m_2} \right].$$

Conditions U_1 , U_2 imply that $\nu_1 | m_2$, $m_2 | m_1 = p^{e_1} \nu_1$, hence $m_2 = \nu_1 p^\epsilon$, $\epsilon \leq e_1$.

If $\nu_1 = 1$, an immediate consequence is that $m_2 \geq p$. Moreover, combining with $(**)$, we get $p^{e_1} \geq 5$ or $p^{e_1} = p^\epsilon = 4$; but the latter case gives no problems since then

$$(***) \quad P_n \geq f_n := 1 - n + \left[\frac{n}{2} \right] + \left[\frac{3n}{4} \right] = f_s + k, \quad n = 4k + s, \quad 0 \leq s \leq 3,$$

$$f_s = 1, 0, 1, 1, \quad s = 0, 1, 2, 3.$$

We treat the several cases:

- If $p \geq 5$, then $m_2 \geq 5$, hence $P_n \geq f_n := 1 - n + [3n/5] + [4n/5]$. Writing $n = 5k + s$ with $0 \leq s \leq 4$, we get

$$P_n \geq f_n = 2k + f_s, \quad f_s = 1, 0, 1, 1, 2, \quad s = 0, 1, 2, 3, 4.$$

Therefore we have $P_n \geq 1$ for $n \geq 2$, and $P_n \geq 2$ for $n \geq 4$.

- If $p = 3$, then $e_1 \geq 2$ and $m_2 \geq 3$. It follows that $P_n \geq f_n := 1 - n + [7n/9] + [2n/3]$. Writing $n = 3k + s$ with $0 \leq s \leq 2$, we get

$$P_n \geq 1 - 3k - s + 2k + [(3k + 7s)/9] + 2k + [2s/3]$$

$$= 1 + k + [(3k + 7s)/9] + [2s/3] - s.$$

Hence $P_n \geq 1 + k$ except for the case $k = 0, s = 1$, which implies that $P_n \geq 1$ for $n \geq 2$ and $P_n \geq 2$ for $n \geq 3$.

- If finally $p = 2$, observe that $e_1 \geq 3$ and $m_2 \geq 2$, hence we have $P_n \geq f_n := 1 - n + [3n/4] + [n/2]$, a case which was already treated in $(***)$.

Assume now $\nu_1 \geq 2$.

- If $p^{e_1} \geq 4$, we have that $P_n \geq 1 - n + [5n/8] + [n/2]$. Writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$P_n \geq 1 - 2k - s + k + [(2k + 5s)/8] + k$$

$$= 1 + [(2k + 5s)/8] - s.$$

It follows that $P_n \geq 1 + [k/4]$ for $s = 0$ and $P_n \geq [(2k + 5)/8]$ for $s = 1$. For the worst case where $p^{e_1} = 4$, $\nu_1 = m_2 = 2$ (this case does not actually

occur since then condition U_1 fails), we have that $P_1 = P_3 = 0$, $P_2 = P_4 = P_5 = P_6 = P_7 = 1$, $P_8 = 2$ and $P_n \geq 2$ for $n \geq 12$.

- If $p^{e_1} = 3$, we cannot have $m_2 = \nu_1 = 2$, since this would contradict inequality $(**)$. Hence we have either $m_2, \nu_1 \geq 3$ or $\nu_1 = 2, m_2 = 6$. We obtain in the respective cases that

$$(*1) \ P_n \geq 1 - n + [5n/9] + [2n/3]$$

resp.

$$(*2) \ P_n \geq 1 - n + [n/2] + [5n/6].$$

For $(*1)$, writing $n = 3k + s$ with $0 \leq s \leq 2$, we get

$$P_n \geq 1 + [(6k + 5s)/9] + [2s/3] - s,$$

which implies that $P_n \geq 1 + [2k/3]$ for $s = 0$, $P_n \geq [(6k + 5)/9]$ for $s = 1$ and $P_n \geq 1 + [(6k + 1)/9]$ for $s = 2$. Hence $P_n \geq 1$ for $n \geq 2$, $P_6 \geq 2$, and $P_n \geq 2$ for $n \geq 8$.

For $(*2)$, writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$P_n \geq 1 + [(4k + 5s)/6] - s,$$

it follows that $P_n \geq 1 + [2k/3]$ for $s = 0$ and $P_n \geq [(4k + 5)/6]$ for $s = 1$. We see that $P_n \geq 1$ for $n \geq 2$, and $P_n \geq 2$ for $n \geq 4$.

- If $p^{e_1} = 2$, we have either $m_2 = \nu_1, \nu_1 \geq 4$ or $m_2 = 2\nu_1, \nu_1 \geq 3$. It follows that

$$(*3) \ P_n \geq 1 - n + [3n/8] + [3n/4]$$

resp.

$$(*4) \ P_n \geq 1 - n + [n/3] + [5n/6].$$

For $(*3)$, writing $n = 4k + s$ with $0 \leq s \leq 3$, we get

$$P_n \geq 1 + [(4k + 3s)/8] + [3s/4] - s,$$

which equals $1 + [k/2]$ for $s = 0$, $[(4k + 3)/8]$ for $s = 1$, $[(4k + 6)/8]$ for $s = 2$, and $1 + [(4k + 1)/8]$ for $s = 3$. Hence for the worst numerical case $\nu_1 = m_2 = 4$ (this case does not actually occur since again condition U_1 fails), we have $P_3 = P_4 = P_6 = P_7 = 1$, $P_2 = P_5 = 0$, $P_8 = 2$, $P_{12} = 2$, $P_{13} = 1$, and $P_n \geq 2$ for $n \geq 14$.

For $(*4)$, writing $n = 3k + s$ with $0 \leq s \leq 2$, we get

$$P_n \geq 1 + [(3k + 5s)/6] - s,$$

hence $P_n \geq 1 + [k/2]$ for $s = 0$, $\geq [(3k + 5)/6]$ for $s = 1$, and $\geq [(3k + 4)/6]$ for $s = 2$. We conclude that $P_n \geq 1$ for $n \geq 3$, $P_6 \geq 2$, and $P_n \geq 2$ for $n \geq 9$.

□

Case (4): Here $K_S \equiv -2F + \sum_i^r (m_i - 1)F'_i$, since $t = 0$ implies that there are no wild fibres.

In view of Theorem 3.12 this situation is exactly as in the classical case. But our main theorem is new also in the classical case, so we proceed to treat case (4).

We assume wlog that

$$m_1 \leq m_2 \leq \dots \leq m_r,$$

and we recall that

$$(***) P_n = \max(0, 1 - 2n + \sum_j [\frac{n(m_j - 1)}{m_j}]).$$

For $r \geq 5$ we have $P_n \geq 1 - 2n + 5[n/2]$, and writing $n = 2k + s$, we get $P_n \geq 1 - 4k - 2s + 5k = 1 + k - 2s$, which is at least 1 for $n \geq 4$, and ≥ 2 for $n \geq 6$.

Assume $r = 4$ and observe once more that the right hand side of $(***)$ is an increasing function of the multiplicities m_j , hence the worst case is $(2, 2, 3, 3)$. Indeed, the worst case would be numerically $(2, 2, 2, 3)$, but indeed this case does not occur, since property U_4 is not fulfilled.

Hence the estimate

$$P_n \geq 1 - 2n + 2[n/2] + 2[2n/3] = 1 + (2[n/2] - n) + (2[2n/3] - n).$$

For even numbers $n = 2k$, we get $P_n = 1 + 2[k/3]$, which is ≥ 1 , and ≥ 3 as soon as $n \geq 6$. For odd numbers $n = 2k + 1$ we get

$$P_n = 2[\frac{4k+2}{3}] - 2k - 1 = 2[\frac{k+2}{3}] - 1,$$

which is ≥ 1 for $n \geq 3$, ≥ 3 as soon as $n \geq 9$.

In the case $r = 3$ recall that conditions U_1, U_2, U_3 are equivalent to the condition that m_k divides the least common multiple of m_i, m_j for each choice of $\{i, j, k\} = \{1, 2, 3\}$.

Assume now $m_1 \geq 4$: by monotonicity the worst case is $(4, 4, 4)$, where, setting $n = 4k + s, 0 \leq s \leq 3$,

$$P_n \geq 3[3n/4] - 2n + 1 = 3(3k + [3s/4]) - 8k - 2s + 1 = 1 + k + 3[3s/4] - 2s.$$

We get

- $k + 1$ for $s = 0, 3$
- k for $s = 2$
- $k - 1$ for $s = 1$.

Hence $P_3 = 1, P_4 = 2, P_n \geq 2$ for $n \geq 10$.

Assume now that $m_1 = 3$. Then 3 divides $LCM(m_2, m_3)$ hence either $m_2 = 3a$ or 3 does not divide m_2 and $m_3 = 3b$. or both alternatives hold.

Keeping in mind the positivity of K_S , equivalent here to $\sum_j \frac{1}{m_j} < 1$, each alternative leads to a worst possible case, i.e. one maximizing $\sum_j \frac{1}{m_j} < 1$.

- (1) $(3, 3a, 3b)$, $a|b, b|a \Rightarrow a = b \Rightarrow (3, 3a, 3a)$: worst case $(3, 6, 6)$;
- (2) $(3, c, 3b)$ c not divisible by 3, $3b|3c, c|b \Rightarrow b = c \Rightarrow (3, c, 3c)$: worst case $(3, 4, 12)$;
- (3) $(3, 3a, c)$ c not divisible by 3, $3a|3c, c|a \Rightarrow a = c \Rightarrow (3, 3a, a)$: same case as the previous.

Recall the plurigenus formula, here it gives respectively

$$(3, 6, 6) : P_n = \max(0, F(n)), F(n) := 1 - 2n + [\frac{2n}{3}] + 2[\frac{5n}{6}].$$

$$(3, 4, 12) : P_n = \max(0, F(n)), F(n) := 1 - 2n + \left[\frac{2n}{3}\right] + \left[\frac{3n}{4}\right] + \left[\frac{11n}{12}\right].$$

In the former case, writing $n = 6k + s$, $0 \leq s \leq 5$, we get

$$F(n) = 2k + F(s), F(s) = 1, -1, 0, 1, 1, 2 (s = 0, 1, \dots, 5)$$

hence $P_3 \geq 1, P_5 \geq 2, P_n \geq 2$ for $n \geq 8$.

In the latter case, writing $n = 12k + s$, $0 \leq s \leq 11$, we get

$$F(n) = 4k + F(s), F(0) = 1, s \geq 1 \Rightarrow F(s) = -s + \left[\frac{2s}{3}\right] + \left[\frac{3s}{4}\right]$$

$$F(s) = 1, -1, 0, 1, 1, 1, 2, 2, 3, 3, 3, 4, 0 \leq s \leq 11,$$

hence $P_n \geq 1$ for $n \geq 3, P_n \geq 2$ for $n \geq 6$.

Assume finally $m_1 = 2$. Then one of m_2, m_3 is even. If $m_j = c$ is odd, $m_i = 2b$, then $c|b, 2b|2c \Rightarrow b = c$, hence we get $(2, b, 2b)$ and the worst case is the case $(2, 5, 10)$ which was already considered in remark 3.4.

Similarly, in case $(2, 2a, 2b)$ again $a = b$, hence we get a triple $(2, 2a, 2a)$ and the worst case is the case $(2, 6, 6)$ which was already considered in remark 3.4.

Remark 4.2. *Our analysis allows us also to see (we omit further details) which are the possible cases where the estimates are sharp in the main theorem.*

- (2): $P_1 = P_2 = P_3 = 0$ exactly in case (4) for triples $(2, b, 2b)$, $b \geq 5, 2 \nmid b$, or $(2, 2a, 2a)$, $a \geq 3$.
- (3): $P_n \leq 1$ for $n \leq 7$ in case (4) for the triple $(2, 5, 10)$ and possibly in case (2) with one wild fibre and $p = \nu = e = 2$.
- (4) : $P_{13} = 1$ exactly in case (4) for the triple $(2, 6, 6)$.

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